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Microscopic derivation of electromagnetic force density in magnetic dielectric media

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Abstract. Macroscopic force density imposed on a linear isotropic magnetic dielectric medium by an arbitrary electromagnetic field is derived by spatially averaging the microscopic Lorentz force density. The obtained expression differs from the commonly used expressions, but the energy-momentum tensor derived from it corresponds to a so-called Helmholtz tensor written for a medium that obeys the Clausius–Mossotti law. Thus, our microscopic derivation unambiguously proves the correctness of the Helmholtz tensor for such media. Also, the expression for the momentum density of the field obtained in our theory is different from the expressions obtained by Minkowski, Abraham, Einstein and Laub, and others. We apply the theory to particular examples of static electric, magnetic and stationary electromagnetic phenomena, and show its agreement with experimental observations. We emphasize that in contrast to a widespread belief the Abraham–Minkowski controversy cannot be resolved experimentally because of incompleteness of the theories introduced by Abraham and Minkowski.

Although the first well-known theoretical model describing electromagnetic forces in ponderable media has been proposed already in 1881 by H von Helmholtz [1], there is still no common agreement on a generally correct theory. In 1908, H Minkowski derived his famous expression for the force density in terms of the electromagnetic energy-momentum tensor and momentum density of the field [2], and almost immediately two alternative models, one

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written by A Einstein and J Laub in 1908 [3], and the other introduced by M Abraham in 1909 [4], appeared to contradict Minkowski's result. Minkowski's and Abraham's expressions are incomplete, because they cannot be used to calculate electro- and magneto-strictive forces [5]. However, the absence of a single commonly accepted theory is traditionally referred to as the Abraham–Minkowski controversy. The Einstein–Laub theory is intended to include the electrostriction and magnetostriction effects, but it turned out to be in disagreement with many experiments.

By using the Helmholtz force density, which in fact was originally written for time-independent macroscopic fields, it is possible to write the energy-momentum tensor in the form of a corrected Minkowski tensor, where two correction terms are added to describe the strictive forces [5] (see also [6, 7]). The field momentum density in this picture has the same form as in Abraham's picture. As far as we know, there have been no experimental results that directly contradict the Helmholtz tensor. Nevertheless, being motivated by the unresolved Abraham–Minkowski controversy, many other expressions for the energy-momentum tensor have been proposed during the last decades [8]–[15] and many experimental attempts to find out which of the expressions are correct have been made [16]–[25]. The experimental results are often treated without paying any attention to the existing Helmholtz tensor. Therefore, they alternately testify in favor of one of the two qualitatively and quantitatively disagreeing theories of Abraham and Minkowski, thus keeping the controversy alive. It is remarkable that both these theories are inherently relativistic, which makes them more popular than Helmholtz's one. However, the majority of practical situations do not deal with relativistic effects, while the uncertainty in the choice between the two contradicting theories restrains their practical application. The Helmholtz tensor, on the other hand, contains density derivatives of relative permittivity ϵ and permeability μ of the medium, which are rather difficult to directly measure even for liquids. However, when using the Clausius–Mossotti relation [26], one can replace these derivatives with simple expressions containing only the parameters ϵ and μ . Note that the Clausius–Mossotti relation well describes gaseous, liquid and many solid-state substances, especially if they have cubic lattice symmetry [27].

The models described above have been derived from energy principles within the macroscopic picture. There have also been several theoretical works on the microscopic derivation of force density [9]–[12], [28, 29], since the results obtained in this way would have a more fundamental physical basis (the real fields in the medium are the microscopic ones, and the measurable macroscopic fields are their mathematical averages). However, on the microscopic level the field–matter interaction picture can easily become too complicated, especially if one wishes to start with the microscopic deformations in the material and then link them to the macroscopic parameters ϵ and μ . Consequently, the theory can lose in its physical insight and practical value.

In this work, we derive the macroscopic electromagnetic force density in a linear isotropic magnetic dielectric medium, starting with a simple microscopic interaction picture, and obtain the result that disagrees with the other known expressions for the force density. We show, however, that for a medium that obeys the Clausius–Mossotti law the derived energy-momentum tensor coincides with the experimentally confirmed Helmholtz tensor. Our general expression for the force density contains contributions from electrostriction and magnetostriction and is not limited to Clausius–Mossotti-type media. This result is essentially what Einstein and Laub wanted to obtain in their theory, according to which deformations and strains should automatically follow from the fundamental laws of electromagnetic forces. Our derivation starts with the microscopic Lorentz force density and is based on the fact that due to Newton's third

law, the own fields of the free and bound charges do not contribute to the force density exerted on them by the total field. Our derivation shows that the knowledge of these fundamental laws of nature is sufficient for also obtaining the forces due to electrostriction and magnetostriction. Our second discovery is a new equation for the momentum density of the field. We would like to stress the observation that this new equation is asymmetrical with respect to ϵ and μ , whereas both the Minkowski and Abraham momenta densities are symmetrical with respect to these quantities. While usually the contribution of the field momentum density to the time-averaged force density is negligibly small, it can play a significant role in the interaction of a strong low-frequency or pulsed electromagnetic field with a dielectric material.

Let us write the microscopic Maxwell's equations and the Lorentz force density as follows:

$$\epsilon_0 \nabla \cdot \mathbf{e} = \xi, \quad (1)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (2)$$

$$-\nabla \times \mathbf{e} = \frac{\partial \mathbf{b}}{\partial t}, \quad (3)$$

$$\nabla \times \mathbf{b} = \epsilon_0 \frac{\partial \mathbf{e}}{\partial t} + \mathbf{j}, \quad (4)$$

$$\mathbf{f}_{\text{mic}} = \xi \mathbf{e} + \mathbf{j} \times \mathbf{b}. \quad (5)$$

Here \mathbf{e} and \mathbf{b} are microscopic electric and magnetic fields in the medium, \mathbf{f}_{mic} is the microscopic Lorentz force density, and the microscopic electric charge and current densities ξ and \mathbf{j} , respectively, are given by

$$\xi = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i), \quad (6)$$

$$\mathbf{j} = \sum_i q_i \frac{\partial \mathbf{r}_i}{\partial t} \delta(\mathbf{r} - \mathbf{r}_i). \quad (7)$$

The electric charges, defined by q_i and having coordinates \mathbf{r}_i , can be divided into free and bound charges, \tilde{q}_j and q_k , respectively. The bound charges form localized groups that belong to individual molecules in the medium. For these groups of charges, we use a dipole approximation that is obtained by expanding the charge and current densities of each molecule into Taylor series around the center of mass of the molecule and truncating the series after the terms containing the molecular electric and magnetic dipole moments \mathbf{d}_l and \mathbf{m}_l , respectively [30]. This is often used when deriving the macroscopic Maxwell's equations from the microscopic ones. Within this approximation, the charge and current densities are

$$\xi = \sum_j \tilde{q}_j \delta(\mathbf{r} - \mathbf{r}_j) - \sum_l \mathbf{d}_l \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_l), \quad (8)$$

$$\mathbf{j} = \sum_j \tilde{q}_j \frac{\partial \mathbf{r}_j}{\partial t} \delta(\mathbf{r} - \mathbf{r}_j) + \sum_l \frac{\partial \mathbf{d}_l}{\partial t} \delta(\mathbf{r} - \mathbf{r}_l) + \sum_l \nabla \times \mathbf{m}_l \delta(\mathbf{r} - \mathbf{r}_l), \quad (9)$$

and the molecules are treated as point particles with electric and magnetic dipole moments. We note, however, that physically the third term in equation (9) represents current loops originating from rotational motion of electric charges in the molecules [30].

Using equations (8) and (9), we write equation (5) as

$$\mathbf{f}_{\text{mic}} = \mathbf{f}_{\text{mic}}^{(f)} + \mathbf{f}_{\text{mic}}^{(b)}, \quad (10)$$

where the force densities $\mathbf{f}_{\text{mic}}^{(f)}$ and $\mathbf{f}_{\text{mic}}^{(b)}$ experienced by free and bound charges, respectively, are given by

$$\mathbf{f}_{\text{mic}}^{(f)} = \sum_j \left(\tilde{q}_j \mathbf{e}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_j) + \tilde{q}_j \frac{\partial \mathbf{r}_j}{\partial t} \times \mathbf{b}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_j) \right), \quad (11)$$

$$\mathbf{f}_{\text{mic}}^{(b)} = \sum_l \left(-\mathbf{d}_l \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_l) \mathbf{e}(\mathbf{r}) + \frac{\partial \mathbf{d}_l}{\partial t} \delta(\mathbf{r} - \mathbf{r}_l) \times \mathbf{b}(\mathbf{r}) + (\nabla \times \mathbf{m}_l \delta(\mathbf{r} - \mathbf{r}_l)) \times \mathbf{b}(\mathbf{r}) \right). \quad (12)$$

The macroscopic force density $\mathbf{f}^{(f)} \equiv \langle \mathbf{f}_{\text{mic}}^{(f)} \rangle$, where the angle brackets denote spatial averaging over a representative elementary volume δV chosen to have a spherical shape, is found with the aid of integration of $\mathbf{f}_{\text{mic}}^{(f)}$ over this volume:

$$\mathbf{f}^{(f)} = \frac{1}{\delta V} \sum_j \int_{\delta V} \tilde{q}_j \left(\mathbf{e}(\mathbf{r}) + \frac{\partial \mathbf{r}_j}{\partial t} \times \mathbf{b}(\mathbf{r}) \right) \delta(\mathbf{r} - \mathbf{r}_j) d\mathbf{r}. \quad (13)$$

The spherical averaging we use is a common one, since it is simple and most general (in a sense that it does not discriminate between different directions in space). From now on we consider the free charges to be identical and drop the subindex j from \tilde{q}_j . The integration property of the delta function yields

$$\mathbf{f}^{(f)} = \frac{\tilde{q}}{\delta V} \sum_{j \text{ in } \delta V} \left(\mathbf{e}(\mathbf{r}_j) + \frac{\partial \mathbf{r}_j}{\partial t} \times \mathbf{b}(\mathbf{r}_j) \right). \quad (14)$$

After the integration, only those indices j that cite the free charges in δV survive in the sum. Since δV is small, we can assume that the free charges are distributed uniformly in δV and move at the same speed $\partial \mathbf{r}_j / \partial t = \mathbf{v}$ (this speed can be equated to the average speed of free charges in δV). In this case, equation (14) can be written in terms of the fields $\mathbf{e}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ averaged over the coordinates of the charges, i.e. $\tilde{N}_{\delta V}^{-1} \sum_{j \text{ in } \delta V} \mathbf{e}(\mathbf{r}_j)$ and $\tilde{N}_{\delta V}^{-1} \sum_{j \text{ in } \delta V} \mathbf{b}(\mathbf{r}_j)$, where $\tilde{N}_{\delta V}$ is the number of charges in δV . Since the charges are assumed to be distributed uniformly, this averaging is equivalent to the volume averaging of $\mathbf{e}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ over δV . We note that the own electric and magnetic fields of the free charges exhibit strong inhomogeneities around each $\mathbf{r} = \mathbf{r}_j$, but exactly at $\mathbf{r} = \mathbf{r}_j$ they are zero⁴, so that the fields $\mathbf{e}(\mathbf{r}_j)$ and $\mathbf{b}(\mathbf{r}_j)$ are the same as without the charge j . In other words, the fields $\mathbf{e}(\mathbf{r}_j)$ and $\mathbf{b}(\mathbf{r}_j)$ are independent of the charge j . The spatial averages of the own fields of the free charges are zero as well, as long as these charges are considered to be distributed uniformly in δV (see section 2.13 in [31] or the text above equation (27) in the present paper). For the volume averages of the fields, we have $\langle \mathbf{e}(\mathbf{r}) \rangle = \mathbf{E}$ and $\langle \mathbf{b}(\mathbf{r}) \rangle = \mathbf{B}$, with \mathbf{E} and \mathbf{B} being macroscopic electric and magnetic fields, respectively. Thus, the force density $\mathbf{f}^{(f)}$ is

$$\mathbf{f}^{(f)} = \frac{\tilde{q}}{\delta V} \tilde{N}_{\delta V} [\langle \mathbf{e}(\mathbf{r}) \rangle + \mathbf{v} \times \langle \mathbf{b}(\mathbf{r}) \rangle] = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (15)$$

⁴ Both electric and magnetic field vectors of a non-relativistic moving electric charge have inversion symmetry with respect to the coordinate of the charge (see, e.g. equations (6-33) and (6-34) in [31]) and any realistic field with this symmetry is zero in the inversion center.

where $\rho = \tilde{q}\tilde{N}_{\delta V}/\delta V$ and $\mathbf{J} = \tilde{q}\tilde{\mathbf{v}}\tilde{N}_{\delta V}/\delta V$ are the macroscopic charge and current densities, respectively. The obtained quantities ρ , \mathbf{J} , \mathbf{E} and \mathbf{B} are considered to have a coordinate at the center of δV . Equation (15) is a standard equation for macroscopic force density imposed by electric and magnetic fields on free charges and currents.

The macroscopic force density due to bound charges, $\mathbf{f}^{(b)} \equiv \langle \mathbf{f}_{\text{mic}}^{(b)} \rangle$, is calculated by us as a sum of three terms, $\mathbf{f}_1^{(b)}$, $\mathbf{f}_2^{(b)}$ and $\mathbf{f}_3^{(b)}$, which are

$$\mathbf{f}_1^{(b)} = - \sum_l \frac{1}{\delta V} \int_{\delta V} \mathbf{d}_l \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_l) \mathbf{e}(\mathbf{r}) d\mathbf{r} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \mathbf{d}_l \cdot \nabla \mathbf{e}(\mathbf{r}_l), \quad (16)$$

$$\mathbf{f}_2^{(b)} = \sum_l \frac{1}{\delta V} \int_{\delta V} \frac{\partial \mathbf{d}_l}{\partial t} \delta(\mathbf{r} - \mathbf{r}_l) \times \mathbf{b}(\mathbf{r}) d\mathbf{r} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \frac{\partial \mathbf{d}_l}{\partial t} \times \mathbf{b}(\mathbf{r}_l), \quad (17)$$

$$\begin{aligned} \mathbf{f}_3^{(b)} &= \sum_l \frac{1}{\delta V} \int_{\delta V} (\nabla \times \mathbf{m}_l \delta(\mathbf{r} - \mathbf{r}_l)) \times \mathbf{b}(\mathbf{r}) d\mathbf{r} \\ &= \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \nabla(\mathbf{m}_l \cdot \mathbf{b}(\mathbf{r}_l)). \end{aligned} \quad (18)$$

In obtaining the result in equation (18), we have used the fact that $\nabla \cdot \mathbf{b} = 0$. Using equation (3), we can rewrite equation (17) as

$$\mathbf{f}_2^{(b)} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \left(\frac{\partial}{\partial t} (\mathbf{d}_l \times \mathbf{b}(\mathbf{r}_l)) + \mathbf{d}_l \times (\nabla \times \mathbf{e}(\mathbf{r}_l)) \right), \quad (19)$$

from which it follows that

$$\mathbf{f}_1^{(b)} + \mathbf{f}_2^{(b)} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \left(\nabla(\mathbf{d}_l \cdot \mathbf{e}(\mathbf{r}_l)) + \frac{\partial}{\partial t} (\mathbf{d}_l \times \mathbf{b}(\mathbf{r}_l)) \right). \quad (20)$$

The overall force density $\mathbf{f}^{(b)}$ is

$$\mathbf{f}^{(b)} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \left(\nabla(\mathbf{d}_l \cdot \mathbf{e}(\mathbf{r}_l)) + \nabla(\mathbf{m}_l \cdot \mathbf{b}(\mathbf{r}_l)) + \frac{\partial}{\partial t} (\mathbf{d}_l \times \mathbf{b}(\mathbf{r}_l)) \right). \quad (21)$$

For molecules within δV and an arbitrary coordinate \mathbf{r} in δV , we can write $\mathbf{e}(\mathbf{r}) = \mathbf{e}_{\text{ext}}(\mathbf{r}) + \mathbf{e}_{\text{own}}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r}) = \mathbf{b}_{\text{ext}}(\mathbf{r}) + \mathbf{b}_{\text{own}}(\mathbf{r})$, where $\mathbf{e}_{\text{own}}(\mathbf{r})$ and $\mathbf{b}_{\text{own}}(\mathbf{r})$ are the fields created by the molecules themselves and the fields $\mathbf{e}_{\text{ext}}(\mathbf{r})$ and $\mathbf{b}_{\text{ext}}(\mathbf{r})$ are external with respect to the molecules within δV . The external fields are independent of the coordinates of the molecules in δV , while the own fields are strongly inhomogeneous around each \mathbf{r}_l . Substituting these expansions into equation (21), we obtain

$$\mathbf{f}^{(b)} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \left(\nabla(\mathbf{d}_l \cdot \mathbf{e}_{\text{ext}}(\mathbf{r}_l)) + \nabla(\mathbf{m}_l \cdot \mathbf{b}_{\text{ext}}(\mathbf{r}_l)) + \frac{\partial}{\partial t} (\mathbf{d}_l \times \mathbf{b}_{\text{ext}}(\mathbf{r}_l)) \right) + \mathbf{f}_{\text{own}}^{(b)}, \quad (22)$$

where

$$\mathbf{f}_{\text{own}}^{(b)} = \frac{1}{\delta V} \sum_{l \text{ in } \delta V} \left(\nabla(\mathbf{d}_l \cdot \mathbf{e}_{\text{own}}(\mathbf{r}_l)) + \nabla(\mathbf{m}_l \cdot \mathbf{b}_{\text{own}}(\mathbf{r}_l)) + \frac{\partial}{\partial t} (\mathbf{d}_l \times \mathbf{b}_{\text{own}}(\mathbf{r}_l)) \right) \quad (23)$$

is equal to zero due to Newton's third law. Indeed, besides the factor of δV^{-1} , $\mathbf{f}_{\text{own}}^{(b)}$ is the total force imposed on the molecules in δV by the fields produced by the molecules themselves.

Removing $\mathbf{f}_{\text{own}}^{(b)}$ from equation (22) and assuming that in δV the molecules have the same dipole moments \mathbf{d} and \mathbf{m} , owing to the smallness of δV , we obtain

$$\mathbf{f}^{(b)} = \frac{1}{\delta V} \sum_{k=x,y,z} \left(d_k \nabla \sum_{l \in \delta V} e_{\text{ext},k}(\mathbf{r}_l) + m_k \nabla \sum_{l \in \delta V} b_{\text{ext},k}(\mathbf{r}_l) \right) + \frac{\partial}{\partial t} \left(\frac{\mathbf{d}}{\delta V} \times \sum_{l \in \delta V} \mathbf{b}_{\text{ext}}(\mathbf{r}_l) \right), \quad (24)$$

where the quantities with subindex k are the Cartesian vector components of the corresponding vector quantities. In the small δV , the molecules can to first order be considered to be distributed uniformly. Taking into account the fact that $\mathbf{e}_{\text{ext}}(\mathbf{r})$ and $\mathbf{b}_{\text{ext}}(\mathbf{r})$ are the same as without the molecules, we can substitute the averaging of the fields over the coordinates of the molecules with volume averaging, which results in

$$\mathbf{f}^{(b)} = \sum_{k=x,y,z} (P_k \nabla E_{\text{ext},k} + M_k \nabla B_{\text{ext},k}) + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}_{\text{ext}}), \quad (25)$$

where $\mathbf{E}_{\text{ext}} = \langle \mathbf{e}_{\text{ext}}(\mathbf{r}) \rangle$ and $\mathbf{B}_{\text{ext}} = \langle \mathbf{b}_{\text{ext}}(\mathbf{r}) \rangle$, and the electric polarization \mathbf{P} and magnetization \mathbf{M} are given by $\mathbf{P} = N_{\delta V} \mathbf{d} / \delta V$ and $\mathbf{M} = N_{\delta V} \mathbf{m} / \delta V$, respectively, with $N_{\delta V}$ denoting the number of molecules in δV . Equation (25) is one of our key results. It obviously differs from the other commonly used expressions for the macroscopic force density on the bound charges [5]–[15], [32]. On the other hand, besides the term $\sum_k M_k \nabla B_{\text{ext},k}$, equation (25) coincides in its form with equation (18) in [33], where the fields \mathbf{E} and \mathbf{B} are used instead of the external fields. The physical principles that have led to equation (25) have in fact been discussed by Brevik [5] (see 148), but neither Brevik nor Hakim, to whose paper [34] Brevik refers, have introduced this equation. The total macroscopic force density is thus

$$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} + \sum_{k=x,y,z} (P_k \nabla E_{\text{ext},k} + M_k \nabla B_{\text{ext},k}) + \frac{\partial}{\partial t} (\mathbf{P} \times \mathbf{B}_{\text{ext}}). \quad (26)$$

The averaged external electric field can be found as $\mathbf{E}_{\text{ext}} = \mathbf{E} - \langle \mathbf{e}_{\text{own}} \rangle$. For an arbitrary charge distribution in the spherical volume δV , the field $\langle \mathbf{e}_{\text{own}} \rangle$ is calculated in a straightforward manner to obtain $\langle \mathbf{e}_{\text{own}} \rangle = -\mathbf{D}_{\delta V} / (3\epsilon_0 \delta V)$, where $\mathbf{D}_{\delta V} = \mathbf{P} \delta V$ is the total dipole moment of the medium within δV (section 2.13 in [31]). Therefore, the external field is

$$\mathbf{E}_{\text{ext}} = \mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0} = \frac{\epsilon + 2}{3} \mathbf{E}, \quad (27)$$

where the medium is assumed to be linear, so that $\mathbf{P} = \epsilon_0(\epsilon - 1)\mathbf{E}$. The obtained field \mathbf{E}_{ext} is equal to the traditional *local field* with the Lorentz correction, which is explained by the fact that both the Lorentz sphere and our δV have spherical shapes.

Similarly, the external magnetic field is calculated as $\mathbf{B}_{\text{ext}} = \mathbf{B} - \langle \mathbf{b}_{\text{own}} \rangle$. Note that while $\langle \mathbf{e}_{\text{own}} \rangle$ is directed oppositely to \mathbf{E} , the field $\langle \mathbf{b}_{\text{own}} \rangle$ being created by electric current loops in the molecules is co-directed with \mathbf{B} . This field is $\langle \mathbf{b}_{\text{own}} \rangle = 2\mu_0 \mathbf{M} / 3$ (see equations (9)–(22) in [31]), which leads to the following equation:

$$\mathbf{B}_{\text{ext}} = \mathbf{B} - \frac{2\mu_0 \mathbf{M}}{3} = \mu_0 \frac{\mu + 2}{3} \mathbf{H}, \quad (28)$$

where the expressions $\mathbf{B} = \mu_0 \mu \mathbf{H}$ and $\mathbf{M} = (\mu - 1)\mathbf{H}$ have been used. It has been shown that for some dense materials the Lorentz correction to the local field is insufficient and, consequently, the Clausius–Mossotti and Lorentz–Lorenz equations are not exact (see e.g. [27]–[35]). In principle, due to similar reasons, the fields \mathbf{E}_{ext} and \mathbf{B}_{ext} can also deviate from those given by equations (27) and (28). In such cases, one should make corrections to these equations.

Substituting equations (27) and (28) into equation (25) and expressing \mathbf{P} and \mathbf{M} through \mathbf{E} and \mathbf{H} as above, we obtain

$$\mathbf{f}^{(b)} = \frac{\epsilon_0(\epsilon - 1)}{3} \left(E^2 \nabla \epsilon + \frac{\epsilon + 2}{2} \nabla E^2 \right) + \frac{\mu_0(\mu - 1)}{3} \left(H^2 \nabla \mu + \frac{\mu + 2}{2} \nabla H^2 \right) + \frac{\partial}{\partial t} \left(\frac{\epsilon_0 \mu_0 (\epsilon - 1)(\mu + 2)}{3} \mathbf{E} \times \mathbf{H} \right). \quad (29)$$

If the medium is non-magnetic, i.e. $\mu = 1$, the third term is equal to Abraham's term $\frac{\partial}{\partial t} \{ \epsilon_0 \mu_0 (\epsilon \mu - 1) \mathbf{E} \times \mathbf{H} \}$ that represents the difference between the Abraham and Minkowski force densities [5]. This term has been proven to exist by the experiments of Walker and Walker [23], in which the material had $\mu = 1$.

The fields \mathbf{E} and \mathbf{H} satisfy the macroscopic Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho, \quad (30)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (31)$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \quad (32)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad (33)$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon \mathbf{E}$ is the electric displacement. Using these equations and combining equations (15) and (29), we find that the total force density $\mathbf{f} = \mathbf{f}^{(f)} + \mathbf{f}^{(b)}$ is described by the following equation:

$$\mathbf{f} = -\nabla \cdot \hat{\mathbf{T}} - \frac{\partial \mathbf{G}}{\partial t}, \quad (34)$$

where the energy-momentum tensor $\hat{\mathbf{T}}$ and the momentum density \mathbf{G} of the field are given by

$$\hat{\mathbf{T}} = \epsilon_0 \left(-\epsilon \mathbf{E} \mathbf{E} + \frac{2 + 2\epsilon - \epsilon^2}{6} E^2 \hat{\mathbf{I}} \right) + \mu_0 \left(-\mu \mathbf{H} \mathbf{H} + \frac{2 + 2\mu - \mu^2}{6} H^2 \hat{\mathbf{I}} \right), \quad (35)$$

$$\mathbf{G} = \frac{2 + 2\epsilon\mu - 2\epsilon + \mu}{3c^2} \mathbf{E} \times \mathbf{H}. \quad (36)$$

In equations (35) and (36), $\mathbf{E} \mathbf{E}$ and $\mathbf{H} \mathbf{H}$ are the outer products of the field vectors, $\hat{\mathbf{I}}$ denotes the unit tensor and c is the speed of light in vacuum. It can be seen that the tensor $\hat{\mathbf{T}}$ is equal to the Helmholtz tensor, when the latter is written for a Clausius–Mossotti medium [5]–[7]. Note that the field momentum density in equation (36) does not appear in the Helmholtz, Minkowski, Einstein–Laub or Abraham pictures. At high frequencies, the permeability μ is equal to 1, and the momentum density \mathbf{G} becomes equal to

$$\mathbf{G}_{\mu=1} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}, \quad (37)$$

which is in agreement with Planck's principle of inertia of energy.

Let us describe some particular examples of application of the obtained equations. As a first example, we consider the well-known experiment on raising a dielectric liquid within a parallel-plate capacitor. The capacitor is partially immersed in the liquid, and the liquid rises when a

horizontal static electric field \mathbf{E} is applied between the plates. According to equation (29), the force density due to the field should have two terms,

$$\mathbf{f}_1 = \epsilon_0(\epsilon - 1)E^2\nabla\epsilon/3, \quad (38)$$

$$\mathbf{f}_2 = \epsilon_0(\epsilon - 1)(\epsilon + 2)\nabla E^2/6. \quad (39)$$

The first term is the force density applied to the surface of the liquid and pushing it down, while the second term is the force density due to the inhomogeneous (fringing) electric field near the edges of the capacitor in the liquid. This second term leads to the elevation of the liquid. Assuming that the size of the capacitor plates is large compared to their separation, the height Δh to which the liquid will rise is calculated from the following equation:

$$\rho_l g \Delta h = \int_{\text{surf}} \mathbf{f}_1 dz + \int_{\text{edge}} \mathbf{f}_2 dz = \frac{\epsilon_0(\epsilon - 1)}{2} E^2, \quad (40)$$

where z is the coordinate along a vertical axis \mathbf{z} drawn in the middle of the capacitor and the integration of \mathbf{f}_1 and \mathbf{f}_2 is performed over regions of inhomogeneous medium at the surface of the liquid and inhomogeneous electric field at the bottom edge of the capacitor, respectively; ρ_l and g are the mass density of the liquid and gravitation acceleration, respectively. We point out that not only \mathbf{f}_2 but also \mathbf{f}_1 is of dipole (gradient) nature. The molecules of the liquid see the field \mathbf{E}_{ext} rather than \mathbf{E} . While \mathbf{E} is continuous across the surface, \mathbf{E}_{ext} is not. The gradient of \mathbf{E}_{ext} results in a dipole force acting on the molecules at the surface and pulling them down. This physical explanation of the surface force is usually missing from the description of the phenomenon. Note that in the Mikowski, Abraham and Einstein–Laub pictures the calculated height Δh is the same as in equation (40), but the reasons for rising of the liquid are different. In both the Minkowski and Abraham pictures, the liquid rises due to an *upward* directed surface force and the volume force due to the fringing electric field in the liquid is zero [5, 36], whereas in the Einstein–Laub picture it is the volume force that raises the liquid and the surface force is zero [5]. The physical explanation given by us comes from equation (25) that is based on the microscopic interaction picture.

It is straightforward to apply the theory to the case of elevation of a magnetic liquid by a horizontal static magnetic field. Owing to the symmetry of tensor $\hat{\mathbf{T}}$, we can immediately write

$$\rho_l g \Delta h = \frac{\mu_0(\mu - 1)}{2} H^2. \quad (41)$$

Both equations (40) and (41) have been verified experimentally [22]. Moreover, these equations can be derived directly from energy principles and are equally correct also for solid materials (see e.g. [37]).

Equations (34) and (35) describe correctly many other experiments. For example, the excess pressure of a dielectric liquid due to a static electric field, \mathbf{E} , has been measured by Hakim and Higham and it has been shown that it satisfies the equation [20]

$$\Delta p = \frac{\epsilon_0(\epsilon - 1)(\epsilon + 2)}{6} E^2, \quad (42)$$

which cannot be obtained within the Abraham, Minkowski and Einstein–Laub pictures but is readily obtained from equations (34) and (35) or from equation (29).

Now let us turn to stationary optical phenomena. Suppose that a plane optical wave is reflected from a perfect mirror surrounded by a transparent dielectric medium. To calculate the

radiation pressure on the mirror, we enclose the reflecting surface of the mirror by two auxiliary surfaces, of which one, A_1 , is chosen inside the mirror where the fields \mathbf{E} and \mathbf{H} are zero, and the other, A_2 , is chosen inside the dielectric but immediately on the surface of the mirror. The time-averaged pressure is calculated by using equations (34) and (35) and applying Gauss's integration law to obtain $\bar{\mathbf{p}} = \bar{\mathbf{T}} \cdot \mathbf{n}_2$, where \mathbf{n}_2 is the unit vector normal to the surface A_2 and pointing inward the mirror and the bar denotes time averaging; the time-averaged momentum density \mathbf{G} is equal to zero. Since at A_2 the electric field is zero, we obtain

$$\bar{\mathbf{p}} = \mathbf{n}_2 \frac{\mu_0(2 + 2\mu - \mu^2)}{6} \overline{H^2}. \quad (43)$$

Writing $\overline{H^2}$ in terms of the incident field intensity I as

$$\overline{H^2} = 4I \sqrt{\epsilon_0 \epsilon / (\mu_0 \mu)} \quad (44)$$

and using the equation $I = \hbar \omega \phi$, where $\hbar \omega$ and ϕ are the photon energy and photon flux density, respectively, we obtain

$$\bar{\mathbf{p}} = \hbar \mathbf{k}_0 \phi \frac{2}{3} \sqrt{\frac{\epsilon}{\mu}} (2 + 2\mu - \mu^2). \quad (45)$$

In this equation, the vector \mathbf{k}_0 is the wavevector in vacuum chosen to point along the propagation direction of the incident wave. If we assume that the mirror receives an average momentum of $2\mathbf{p}_{\text{ph}}$ per reflected photon, then the calculated pressure $\bar{\mathbf{p}}$ must be equal to $2\mathbf{p}_{\text{ph}}\phi$, from which we find

$$\mathbf{p}_{\text{ph}} = \hbar \mathbf{k}_0 \frac{1}{3} \sqrt{\frac{\epsilon}{\mu}} (2 + 2\mu - \mu^2) = n \hbar \mathbf{k}_0, \quad (46)$$

where n is the index of refraction of the medium. The last equality in equation (46) is obtained after setting μ to 1. Obviously, in the considered case the average photon momentum \mathbf{p}_{ph} in the medium is equal to the photon momentum in the Minkowski picture. It is worth mentioning that this momentum has been obtained in experimental studies of radiation pressure not only on a mirror in a dielectric medium [16, 17] but also on atoms in a Bose–Einstein condensate [24] and charge carriers in a semiconductor [18]. The fact that each recoiled atom in [24] and charge carrier in [18] experiences the same photon momentum as a mirror in a dielectric is what in our opinion could be expected.

We proceed to the calculation of a radiation pressure imposed by the reflected wave in the above example on the medium. The interference of the incident and reflected waves forms a standing optical wave, and we want to know the pressure on the medium between the mirror and the first interference maximum of the electric field. This maximum occurs on a surface A_3 located at a distance of $\lambda/4$ from the mirror. At this distance the magnetic field of the wave is equal to zero. The pressure is calculated as

$$\bar{\mathbf{p}} = -\bar{\mathbf{T}}_{A_3} \cdot \mathbf{n}_3 - \bar{\mathbf{T}}_{A_2} \cdot \mathbf{n}_2, \quad (47)$$

where the tensors are evaluated on surfaces A_2 and A_3 and \mathbf{n}_3 is directed outward the mirror. Setting μ to 1, we obtain

$$\bar{\mathbf{p}} = \mathbf{n}_3 \frac{\epsilon_0(\epsilon - 1)(\epsilon + 2)}{6} \overline{E_{\lambda/4}^2}, \quad (48)$$

where $E_{\lambda/4}$ is the electric field strength at a distance of $\lambda/4$ from the mirror. This result is in agreement with equation (42). Equation (48) shows that the medium is compressed toward the interference maxima of the electric field of the wave.

It is similarly straightforward to evaluate the force density and radiation pressure in a medium that interacts with a laser beam instead of a plane wave. For example, an ordinary Gaussian laser beam propagating in a linear dielectric medium will compress the medium toward the beam axis. By applying equations (34) and (35), or equation (29), and the fact that $\mu = 1$, it can be shown that the time-averaged compressive force density is given by

$$\bar{\mathbf{f}} = \frac{\epsilon_0(\epsilon - 1)(\epsilon + 2)}{6} \nabla \overline{E^2}, \quad (49)$$

independently of beam polarization. If, on the other hand, the beam is normally incident from vacuum onto a flat surface of a dielectric liquid, then the resulting force density has two components. The first component acts on the surface, pushing it down with a pressure of

$$p_{\text{down}} = \epsilon_0(\epsilon - 1)^2 \overline{E^2} / 6, \quad (50)$$

calculated by taking into account the continuity of \mathbf{E} and \mathbf{H} across the surface. The second component acts inside the liquid and is described by equation (49). The excess hydrostatic pressure due to this second component leads to the rising of liquid surface. For this pressure we can write

$$p_{\text{up}} = \epsilon_0(\epsilon - 1)(\epsilon + 2) \overline{E^2} / 6. \quad (51)$$

The overall pressure that elevates the surface of the liquid is then

$$p = p_{\text{up}} - p_{\text{down}} = \epsilon_0(\epsilon - 1) \overline{E^2} / 2. \quad (52)$$

This result is in agreement with the experiments of Ashkin and Dziedzic [21]. Note that equation (52) can also be obtained by assuming that the average momentum per photon is given by equation (46) and applying the momentum conservation law. However, the question of correctness of existing different expressions for the *real* photon momentum in a medium is not a topic we want to consider here. Furthermore, the classical macroscopic picture that we have used in the above examples is, in our opinion, not quite appropriate for making conclusions about the real photon momentum in a medium. In fact, such conclusions, being made on the basis of different experimental observations, continuously appear in the literature to contradict each other. Even a quantum mechanical description can yield different photon momenta in the same dielectric material under different experimental conditions, if it uses the macroscopic and, therefore, already averaged quantities and operators [38].

Equations (34)–(36) can be applied not only to stationary and static electromagnetic fields but also to such dynamical phenomena, in which the time derivative of the field momentum density plays a significant role. This, however, can be the case only within a short time interval so that the measured, time-averaged force density due to the field momentum is usually close or equal to zero. We nevertheless believe that it is possible to experimentally verify equation (36) by using a low-frequency or pulsed electromagnetic field and a medium with $\mu \neq 1$. In particular, we expect that the difference between equation (36) and other expressions for the field momentum can become evident when dealing with the interaction of strong laser pulses with optical-frequency magnetic materials, such as recently developed metamaterials.

In summary, we have obtained a general expression, equation (26), for the macroscopic force density imposed by an electromagnetic field on a linear isotropic magnetic dielectric

medium by spatially averaging the microscopic Lorentz force density. This equation is an important fundamental result exhibiting the true nature of the macroscopic force density in a medium. We have evaluated the volume-averaged fields produced by sources that are external with respect to the averaging volume and transformed equation (26) into equation (34) written in terms of the energy-momentum tensor and the momentum density of the field. The obtained tensor has been found to be equal to the so-called Helmholtz tensor written for a Clausius–Mossotti medium. By this, our microscopic derivation unambiguously proves the correctness of this tensor and insufficiency of the Abraham and Minkowski tensors. Moreover, different energy-momentum tensors are obtained if media are considered not to satisfy the classical Clausius–Mosotti law. It is important that in our derivation we have used only the microscopic Lorentz force density, Maxwell’s equations and Newton’s third law. Thus we have shown that the knowledge of these fundamental laws is sufficient to also obtain the electro- and magneto-strictive forces. The expression derived by us for the momentum density of the field, equation (36), does not coincide with any other existing expression for this quantity. In particular, we would like to stress that our expression is asymmetrical with respect to ϵ and μ . We anticipate this new expression to attract the attention of experimentalists and find useful applications in the future.

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